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BOUNDARY PROBLEM SOLUTION FOR EIGENMODES IN COAXIAL QUAD-RIDGED WAVEGUIDES

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The boundary problem for eigenmodes in coaxial quad-ridged waveguides has been solved by the transverse field-matching technique. The formulas obtained provide possibilities to calculate cutoff wave numbers and electric and magnetic fields distributions of TEM, TE and TM modes in the presence of the ridges either on the inner or on the outer perfectly conducting cylinder. The analysis of the dependences of cutoff wave numbers and electric field distributions convergences on the number of partial modes has been carried out. It has been shown that for calculation of cutoff wave numbers with residual error less than 0.1 % it is enough to utilize 27 partial modes, and for the correct calculation of fields distributions one should utilize more than 30 partial modes.

Introduction

Ridged structures are widely used in modern waveguide devices. The utilization of ridges enables to create ultrawideband devices, to provide ultrawideband matching of hollow waveguides with coaxial transmission lines and to create discontinuities of required type in narrowband devices. Ridged structures are used in filters [1—3], polarizers [4—6], waveguides [7—10], antennas [11—13], orthomode transducers [14—16], lasers [17—19], resonators [20, 21] and other devices.

For development of the devices based on ridged structures one needs to know modal characteristics of ridged waveguides, namely, eigenmodes cutoff frequencies (or cutoff wave numbers) and their fields distributions. The characteristics of ridged waveguides' eigenmodes for rectangular cross-section have been analyzed in [22—24]. The eigenmodes of square ridged waveguides have been investigated in [9]. The eigenmodes of ridged waveguides for circular cross section have been analyzed in [24]. The eigenmodes of elliptical ridged waveguides have been investigated in [10]. The eigenmodes of rectangular coaxial ridged waveguides have been analyzed in [8]. In [25] authors of this paper have solved the boundary problem for sectoral coaxial ridged waveguides using integral equation technique, and in [26] their eigenmodes have been analyzed.

In this paper the boundary problem solution for the coaxial quad-ridged waveguides (CQRW) has been obtained using transverse field-matching technique and the solutions convergence analysis has been performed for the dependences of cutoff wave numbers and electric field distributions (EFD) on the number of partial modes.

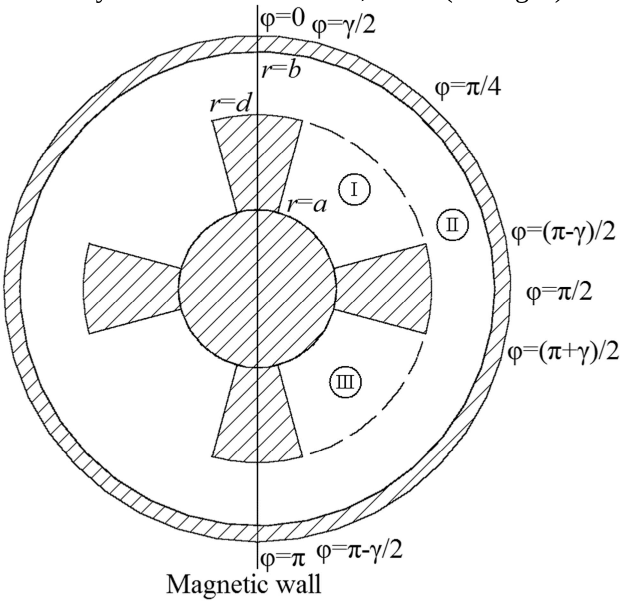
Problem statement

Configurations of hollow infinite CQRW under study and denotations of their cross sectional dimensions are shown in Fig. 1, namely, the CQRW with ridges on the inner perfectly conducting cylinder is depicted in Fig. 1*a*, and the CQRW with ridges on the outer cylinder is shown in Fig. 1*b* (hereinafter referred to as subscripts "in" and "out" respectively).

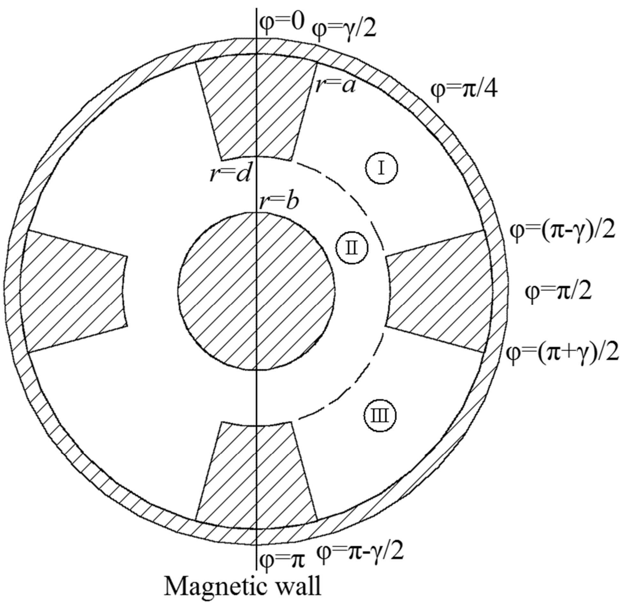
We will investigate only that eigenmodes, for which the CQRW vertical symmetry plane ($\phi = 0$) is a magnetic wall. Due to the symmetry of the CQRW relative to the horizontal plane $\phi = \pi/2$, it is expedient to obtain the boundary problem solution separately for the eigenmodes with antisymmetric and symmetric relative to that plane EFD. Consequently, the fields distributions in the region III will be, respectively, antisymmetric or symmetric relative to the distributions in the region I. The eigenmodes with symmetric relative to the plane $\phi = \pi/2$ EFD can be conventionally divided into two types: 1) the eigenmodes with symmetric EFD relative to the plane $\varphi = \pi/4$, 2) the eigenmodes with antisymmetric EFD relative to the plane $\varphi = \pi/4$. For the eigenmodes with antisymmetric EFD the plane $\varphi = \pi/2$ is the electric wall, and for the eigenmodes with symmetric EFD it is the magnetic wall. Besides, for the eigenmodes with symmetric EFD relative to the plane $\varphi = \pi/2$, that are also antisymmetric relative to the plane $\varphi = \pi/4$, that plane is electric wall, for the eigenmodes, EFD of which are symmetric relative to the plane $\varphi = \pi/4$, it is magnetic wall.

Consequently, for the TE and the TM eigenmodes with antisymmetric EFD the boundary problem should be solved only for the region limited by the magnetic wall $\phi = 0$, by the electric wall $\varphi = \pi/2$ and by perfectly conducting circular cylinders with radii $r = a$, $r = b$ (see Fig.

2). For the eigenmodes with symmetric EFD relative to the plane $\varphi = \pi/2$ the boundary problem should be solved only for the region limited by the magnetic wall $\phi = 0$, and by electric or by magnetic wall $\phi = \pi/4$ (for the eigenmodes, respectively, with antisymmetric or symmetric EFD relative to it) and by perfectly conducting circular cylinders with radii $r = a$, $r = b$ (see Fig. 3).

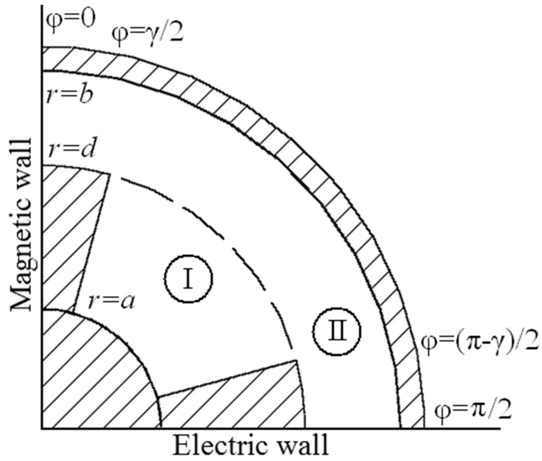


(*a*)

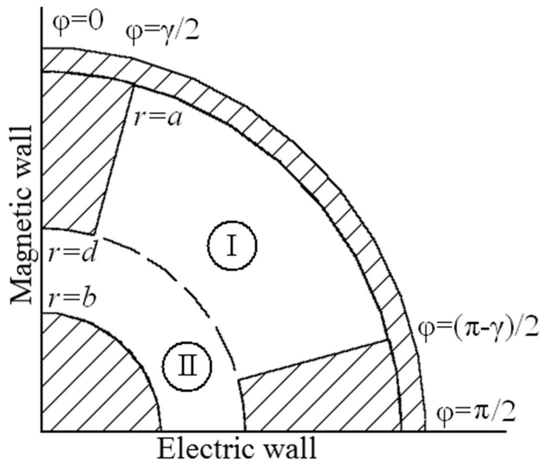


(*b*)

Fig. 1. Cross sections of coaxial quad-ridged waveguides with ridges: (*a*) on inner conducting circular cylinder; (*b*) on outer conducting circular cylinder.



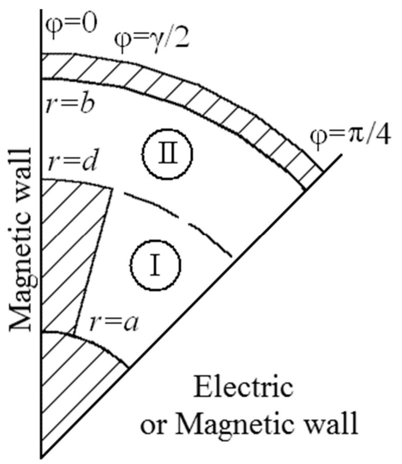
(*a*)



(*b*)

Fig. 2. Computational models of coaxial quad-ridged waveguides with ridges: (*a*) on inner conducting circular cylinder; (*b*) on outer conducting circular cylinder.

EFD of the TEM mode are symmetric relative to not only the planes $\phi = 0$ and $\phi = \pi/2$, but also relative to the planes $\phi = \pi/4$ and $\phi = 3\pi/4$, because all four ridges have the same potential. Therefore for the TEM mode the CQRW has four magnetic walls ($\phi = 0$; $\pi/4$; $\pi/2$; $3\pi/4$). Consequently, for the analysis of TEM eigenmode it is necessary to solve boundary problem only for the region limited by the magnetic walls $\phi = 0$, $\phi = \pi/4$ and by perfectly conducting circular cylinders with radii $r = a$, $r = b$ (see Fig. 3, in which both walls should be chosen as magnetic ones).



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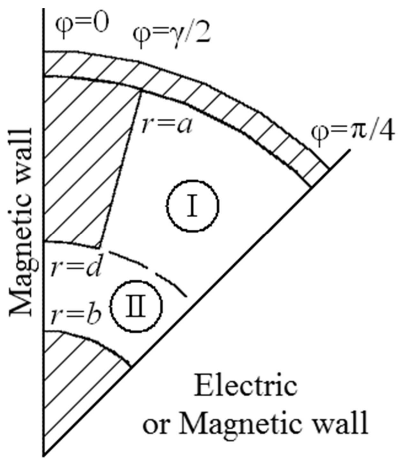

(*b*)

Fig. 3. Computational models of coaxial quad-ridged waveguides taking into account symmetry properties with ridges: (*a*) on inner conducting circular cylinder; (*b*) on outer conducting circular cylinder.

TEM eigenmode

In the regions I and II (Fig. 3, in which both walls should be chosen as magnetic ones) we represent the electric field components E_r and E_ϕ in the form of infinite sums of the partial modes with unknown amplitudes, each of which satisfies the Maxwell equations in the cylindrical coordinate system and boundary conditions at the magnetic walls and at the perfectly conducting surfaces of CQRW:

$$E_r^{\mathrm{I}}(r,\phi)=\sum_{n=0}^{\infty} A_n \sin[l_1(n)(\phi-\gamma/2)]r^{-1+l_1(n)}\,; \quad (1)$$

$$E_r^{\mathrm{II}}(r,\phi)=\sum_{m=0}^{\infty} B_m \cos[l_2(m)\phi]r^{-1-l_2(m)}\,; \quad (2)$$

$$E_\phi^{\mathrm{I}}(r,\phi)=\sum_{n=0}^{\infty} A_n \cos[l_1(n)(\phi-\gamma/2)]r^{-1+l_1(n)}\,; \quad (3)$$

$$E_\phi^{\mathrm{II}}(r,\phi)=\sum_{m=0}^{\infty} B_m \sin[l_2(m)\phi]r^{-1-l_2(m)}\,, \quad (4)$$

where $l_1(n)=2\pi(2n+1)/(\pi-2\gamma)$; $l_2(m)=4m$; A_n, B_m are unknown amplitude coefficients.

The boundary conditions at the interface between the regions I and II (Fig. 3) are as follows:

$$E_r^{\mathrm{II}}(r=d,\ \phi\in[\gamma/2;\ \pi/4])=$$
$$=E_r^{\mathrm{I}}(r=d,\ \phi\in[\gamma/2;\ \pi/4])\,; \quad (5)$$

$$E_\phi^{\mathrm{II}}(r=d,\ \phi\in[\gamma/2;\ \pi/4])=$$
$$=E_\phi^{\mathrm{I}}(r=d,\ \phi\in[\gamma/2;\ \pi/4])\,. \quad (6)$$

Besides, at the perfectly conducting surface of the ridge at $r=d$ and $\phi\in[0;\ \gamma/2]$:

$$E_\phi^{\mathrm{II}}(r=d,\ \phi\in[0;\ \gamma/2])=0\,. \quad (7)$$

Having substituted (1)—(4) in (5)—(7), we obtain:

$$\sum_{m=0}^{\infty} B_m \cos[l_2(m)\phi]d^{-1-l_2(m)}=$$
$$\sum_{n=0}^{\infty} A_n \sin[l_1(n)(\phi-\gamma/2)]d^{-1+l_1(n)}\,,\ \phi\in[\gamma/2;\ \pi/4]; \quad (8)$$

$$\sum_{m=0}^{\infty} B_m \sin[l_2(m)\phi]d^{-1-l_2(m)}=$$
$$\sum_{n=0}^{\infty} A_n \cos[l_1(n)(\phi-\gamma/2)]d^{-1+l_1(n)}\,,\ \phi\in[\gamma/2;\ \pi/4], \quad (9)$$

$$\sum_{m=0}^{\infty} B_m \sin[l_2(m)\phi]d^{-1-l_2(m)}=0\,,\ \phi\in[0;\ \gamma/2]. \quad (10)$$

Multiplying left and right parts of the equation (8) by the system of functions $\sin[l_1(p)(\phi-\gamma/2)]$, $p=0,1,2,...$ and integrating the resulting relation at the interval $[\gamma/2;\ \pi/4]$, at which the system of these functions is orthogonal, we obtain

$$\sum_{m=0}^{\infty} B_m I_1(p,m) d^{-1-l_2(m)}=A_p\frac{\pi-2\gamma}{8}d^{-1+l_1(p)}\,, \quad (11)$$

from whence it follows, that the amplitude of the p-th partial mode in the region I (Fig. 3)

$$A_p=8\sum_{m=0}^{\infty} B_m I_1(p,m)d^{-1-l_2(m)}/[(\pi-2\gamma)d^{-1+l_1(p)}]. \quad (12)$$

The value of $I_1(p,m)$ in the formulas (11), (12) is determined by the relation

$$I_1(p,m)=\int_{\gamma/2}^{\pi/4}\cos[l_2(m)\phi]\sin[l_1(p)(\phi-\gamma/2)]\mathrm{d}\phi .$$

In the same way, the amplitude of the p-th partial mode in the region I (Fig. 3) can be derived from (9) as

$$A_p=8\sum_{m=0}^{\infty}B_m I_2(p,m)d^{-1-l_2(m)}/[(\pi-2\gamma)d^{-1+l_1(p)}], \quad (13)$$

where $I_2(p,m)=\int_{\gamma/2}^{\pi/4}\sin[l_2(m)\phi]\cos[l_1(p)(\phi-\gamma/2)]\mathrm{d}\phi$.

Having equated (12) and (13), we obtain the system of equations

$$\sum_{m=0}^{\infty}B_m I_1(p,m)d^{-1-l_2(m)}=\sum_{m=0}^{\infty}B_m I_2(p,m)d^{-1-l_2(m)} ;$$

$$\sum_{m=0}^{\infty}B_m d^{-1-l_2(m)}\left[I_1(p,m)-I_2(p,m)\right]=0 . \quad (14)$$

Simplifying the expressions (14) yields the result

$$\sum_{m=0}^{\infty}B_m F_1(p,m)=0, \ p=0,1,2,..., \quad (15)$$

where $F_1(p,m)=d^{-1-l_2(m)}\left[I_1(p,m)-I_2(p,m)\right]$.

Next, we multiply left and right parts of the equation (10) by the system of functions $\sin[l_2(q)\phi]$, $q=0,1,2,...$ and integrate the obtained relation at the interval $[0;\ \gamma/2]$. As a result, we get

$$\sum_{m=0}^{\infty}B_m I_3(q,m)d^{-1-l_2(m)}=0 , \quad (16)$$

where $I_3(q,m)=\int_{0}^{\gamma/2}\sin[l_2(m)\phi]\sin[l_2(q)\phi]\mathrm{d}\phi$.

Simplifying the expression (16) yields the following result:

$$\sum_{m=0}^{\infty}B_m F_2(q,m)=0, \ q=0,1,2,... \quad (17)$$

where $F_2(q,m)=I_3(q,m)d^{-1-l_2(m)}$.

Combining the systems of equations (15), (17) and limiting the number of partial modes in the region II, we obtain the following homogeneous system of linear algebraic equations (SLAE) with unknown partial modes amplitudes B_m:

$$\begin{cases}\sum_{m=0}^{M-1}B_m F_1(p,m)=0, & p=0,1,...,(P-1)\\ \sum_{m=0}^{M-1}B_m F_2(q,m)=0, & q=0,1,...,(M-P-1)\end{cases} . \quad (18)$$

At the fixed number of partial modes M , we define the number of equations of the first type by the angular widths ratio of the regions I and II (Fig. 3) [27] as $P=\mathrm{int}\left[(\pi-2\gamma)M/\pi\right]$, where integer part is rounded up or down.

The system of linear algebraic equations (18) can be rewritten in the matrix form:

$$\begin{bmatrix}F_{0,0} & \cdots & F_{0,M-1}\\ \vdots & \ddots & \vdots\\ F_{M-1,0} & \cdots & F_{M-1,M-1}\end{bmatrix}\begin{bmatrix}B_0\\ \vdots\\ B_{M-1}\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix} \quad (19)$$

The elements of the matrix $[F]$ are determined by the following relation

$$F(i,j)=\begin{bmatrix}F_1(i,j), & i=0,\ 1,...,(P-1)\\ F_2((i-P),j), & i=P,\ (P+1),...,(M-1)\end{bmatrix}.$$

The condition of nontrivial solution of the homogeneous SLAE (19) is the equality to zero of the matrix $[F]$ determinant. This condition is satisfied, because the row of the matrix $[F]$ determinant with the index P is zero:

$$F(P,j)=F_2(0,j)=I_3(0,j)d^{-1-l_2(j)}=$$

$$=\int_{0}^{\gamma/2}\sin(4j\phi)\sin(0)\mathrm{d}\phi\cdot d^{-1-l_2(j)}=0 .$$

While solving the homogeneous SLAE (19), the row with the index P must be excluded from the matrix $[F]$.

To solve the homogeneous SLAE (19), let us assume that $B_0=1$. Then, we get:

$$\begin{bmatrix}F_{0,1} & \cdots & F_{0,M-1}\\ \vdots & \ddots & \vdots\\ F_{P-1,1} & \cdots & F_{P-1,M-1}\\ F_{P+1,1} & \cdots & F_{P+1,M-1}\\ \vdots & \ddots & \vdots\\ F_{M-1,1} & \cdots & F_{M-1,M-1}\end{bmatrix}\begin{bmatrix}B_1\\ \vdots\\ B_{M-1}\end{bmatrix}=-\begin{bmatrix}F_{0,0}\\ \vdots\\ F_{P-1,0}\\ F_{P+1,0}\\ \vdots\\ F_{M-1,0}\end{bmatrix};$$

$$\begin{bmatrix} B_1 \\ \vdots \\ B_{M-1} \end{bmatrix} = -\begin{bmatrix} F_{0,1} & \cdots & F_{0,M-1} \\ \vdots & \ddots & \vdots \\ F_{P-1,1} & \cdots & F_{P-1,M-1} \\ F_{P+1,1} & \cdots & F_{P+1,M-1} \\ \vdots & \ddots & \vdots \\ F_{M-1,1} & \cdots & F_{M-1,M-1} \end{bmatrix}^{-1} \begin{bmatrix} F_{0,0} \\ \vdots \\ F_{P-1,0} \\ F_{P+1,0} \\ \vdots \\ F_{M-1,0} \end{bmatrix}.$$

By using these systems of equations, we define all partial mode amplitudes B_m. Then, the partial mode amplitudes A_p are determined by formulas (12) or (13). Further, we find distributions of electric field components in regions I and II (Fig. 3) by the formulas (1)—(4). Electric field components in the whole cross section of CQRW are determined using the EFD symmetry properties of TEM mode. The components of the TEM mode magnetic field can be defined by the formulas: $H_\phi(r,\phi) = E_r(r,\phi)/Z$, $H_r(r,\phi) = -E_\phi(r,\phi)/Z$, where $Z = \sqrt{\mu_a/\varepsilon_a}$ is the mode impedance depending on the CQRW homogeneous filling parameters μ_a , ε_a only.

TE modes antisymmetric relative to the plane $\varphi = \pi/2$

The TE eigenmodes designations introduced in this section coincide with the ones given above for the TEM mode, but they refer only to the TE modes. In the regions I and II (Fig. 2) we represent the fields H_z and E_φ in the form of infinite sums of the partial modes with unknown amplitudes and cutoff wave numbers, each of which satisfies the Maxwell equations in the cylindrical coordinate system and boundary conditions at the magnetic, electric walls and at the perfectly conducting surfaces of CQRW (see Fig. 2):

$$H_z^{\mathrm{I}}(r,\phi) = \sum_{n=0}^{\infty} A_n \cos[l_1(n)(\phi - \gamma/2)] \times$$
$$\times [J'_{l_1(n)}(k_c a) Y_{l_1(n)}(k_c r) - Y'_{l_1(n)}(k_c a) J_{l_1(n)}(k_c r)]; \quad (20)$$

$$H_z^{\mathrm{II}}(r,\phi) = \sum_{m=0}^{\infty} B_m \sin[l_2(m)\phi] \times$$
$$\times [J'_{l_2(m)}(k_c b) Y_{l_2(m)}(k_c r) - Y'_{l_2(m)}(k_c b) J_{l_2(m)}(k_c r)]; \quad (21)$$

$$E_\phi^{\mathrm{I}}(r,\phi) = Z(f, k_{\text{кр}}) \sum_{n=0}^{\infty} A_n \cos[l_1(n)(\phi - \gamma/2)] \times$$
$$\times [J'_{l_1(n)}(k_c a) Y'_{l_1(n)}(k_c r) - Y'_{l_1(n)}(k_c a) J'_{l_1(n)}(k_c r)]; \quad (22)$$

$$E_\phi^{\mathrm{II}}(r,\phi) = Z(f, k_{\text{кр}}) \sum_{m=0}^{\infty} B_m \sin[l_2(m)\phi] \times$$
$$\times [J'_{l_2(m)}(k_c b) Y'_{l_2(m)}(k_c r) - Y'_{l_2(m)}(k_c b) J'_{l_2(m)}(k_c r)], \quad (23)$$

where $l_1(n) = 2\pi n/(\pi - 2\gamma)$; $l_2(m) = 2m+1$; $Z(f, k_c) = 2\pi i f \mu_a / k_c$; A_n, B_m are unknown amplitude coefficients; $J_l(x)$, $Y_l(x)$, $J'_l(x)$, $Y'_l(x)$ are Bessel functions of the first and the second kinds and their derivatives; k_c defines a cutoff mode number; i denotes an imaginary unit; f designates the frequency; μ_a is absolute permeability of CQRW inner medium.

The boundary conditions at the interface between the regions I and II (Fig. 2) are as follows:

$$E_\phi^{\mathrm{II}}(r = d,\ \phi \in [\gamma/2;\ (\pi-\gamma)/2]) =$$
$$= E_\phi^{\mathrm{I}}(r = d,\ \phi \in [\gamma/2;\ (\pi-\gamma)/2]), \quad (24)$$
$$H_z^{\mathrm{II}}(r = d,\ \phi \in [\gamma/2;\ (\pi-\gamma)/2]) =$$
$$= H_z^{\mathrm{I}}(r = d,\ \phi \in [\gamma/2;\ (\pi-\gamma)/2]). \quad (25)$$

On the perfectly conducting surfaces of the ridges at $r = d$ and $\phi \in [0;\ \gamma/2] \cup [(\pi-\gamma)/2;\ \pi/2]$, the following condition is met:

$$E_\phi^{\mathrm{II}}(r = d,\ \phi \in [0;\ \gamma/2] \cup [(\pi-\gamma)/2;\ \pi/2]) = 0 \quad . \quad (26)$$

Having substituted (20)—(23) in (24)—(26), we obtain:

$$\sum_{m=0}^{\infty} B_m \sin[l_2(m)\phi] J'Y'[l_2(m), k_c b, k_c d] =$$
$$= \sum_{n=0}^{\infty} A_n \cos[l_1(n)(\phi - \gamma/2)] J'Y'[l_1(n), k_c a, k_c d],$$
$$\phi \in [\gamma/2;\ (\pi-\gamma)/2]; \quad (27)$$

$$\sum_{m=0}^{\infty} B_m \sin[l_2(m)\phi] J'Y[l_2(m), k_c b, k_c d] =$$
$$= \sum_{n=0}^{\infty} A_n \cos[l_1(n)(\phi - \gamma/2)] J'Y[l_1(n), k_c a, k_c d],$$
$$\phi \in [\gamma/2;\ (\pi-\gamma)/2]; \quad (28)$$

$$\sum_{m=0}^{\infty} B_m \sin[l_2(m)\phi] J'Y'[l_2(m), k_c b, k_c d] = 0,$$
$$\phi \in [0;\ \gamma/2] \cup [(\pi-\gamma)/2;\ \pi/2], \quad (29)$$

where $J'Y(l,x,y) = J'_l(x) Y_l(y) - Y'_l(x) J_l(y)$, $J'Y'(l,x,y) = J'_l(x) Y'_l(y) - Y'_l(x) J'_l(y)$.

Multiplying left and right parts of the equation (27) by the functions $\cos[l_1(p)(\phi - \gamma/2)]$, $p = 0,1,2,\ldots$ and integrating the result at the interval $[\gamma/2;\ (\pi-\gamma)/2]$, at which the system of these functions is orthogonal, we obtain:

$$\sum_{m=0}^{\infty} B_m I_1(p,m) J'Y'[l_2(m), k_c b, k_c d] =$$
$$= A_p(\pi - 2\gamma)/4(1+\delta_{p0}) J'Y'[l_1(p), k_c a, k_c d], \quad (30)$$

whence the amplitude of the p-th partial mode in the region I (Fig. 2) is defined as follows

$$A_p = \frac{4\sum_{m=0}^{\infty} B_m I_1(p,m) J'Y'[l_2(m), k_c b, k_c d]}{(\pi - 2\gamma)(1+\delta_{p0}) J'Y'[l_1(p), k_c a, k_c d]}. \quad (31)$$

In the formulas (30), (31), the following designations are used:

$$I_1(p,m) = \int_{\gamma/2}^{(\pi-\gamma)/2} \sin[l_2(m)\phi]\cos[l_1(p)(\phi - \gamma/2)]d\phi;$$

δ_{p0} is the Kronecker delta.

In the same way, the amplitude of the p-th partial mode in the region I (Fig. 2) can be derived from the expression (28)

$$A_p = \frac{4\sum_{m=0}^{\infty} B_m I_1(p,m) J'Y[l_2(m), k_c b, k_c d]}{(\pi - 2\gamma)(1+\delta_{p0}) J'Y[l_1(p), k_c a, k_c d]}. \quad (32)$$

Having equated (31) and (32), one can obtain

$$\sum_{m=0}^{\infty} B_m I_1(p,m) \{ \frac{J'Y'[l_2(m), k_c b, k_c d]}{J'Y'[l_1(p), k_c a, k_c d]} -$$
$$- \frac{J'Y[l_2(m), k_c b, k_c d]}{J'Y[l_1(p), k_c a, k_c d]} \} = 0. \quad (33)$$

By simplifying the expression (33), we obtain

$$\sum_{m=0}^{\infty} B_m F_1(p, m, k_c a, k_c b, k_c d) = 0, \quad p = 0,1,2,\ldots, \quad (34)$$

where $F_1(p,m,x,y,z) = I_1(p,m)\times$
$$\times \left[\frac{J'Y'(l_2(m), y, z)}{J'Y'(l_1(p), x, z)} - \frac{J'Y(l_2(m), y, z)}{J'Y(l_1(p), x, z)} \right].$$

Multiplying left and right parts of the equation (29) by the system of functions $\sin[l_2(q)\phi]$, $q = 0,1,2,\ldots$ and integrating the resulting relation at the disjunction of intervals $[0;\ \gamma/2] \cup [(\pi-\gamma)/2;\ \pi/2]$, we get

$$\sum_{m=0}^{\infty} B_m I_2(q,m) J'Y'[l_2(m), k_c b, k_c d] = 0, \quad (35)$$

where $$I_2(q,m) = \int_0^{\gamma/2} \sin[l_2(m)\phi]\sin[l_2(q)\phi]d\phi + \int_{(\pi-\gamma)/2}^{\pi/2} \sin[l_2(m)\phi]\sin[l_2(q)\phi]d\phi.$$

Simplifying the expression (35) yields the following result:

$$\sum_{m=0}^{\infty} B_m F_2(q, m, k_c b, k_c d) = 0, \quad q = 0,1,2,\ldots, \quad (36)$$

where $F_2(q,m,y,z) = I_2(q,m) J'Y'[l_2(m), y, z]$.

Having united the systems of equations (34), (36) and limiting the number of partial modes in the region II, we obtain the following homogeneous SLAE with unknown partial mode amplitudes B_m:

$$\begin{cases} \sum_{m=0}^{M-1} B_m F_1(p, m, k_c a, k_c b, k_c d) = 0, & p = 0,1,\ldots,(P-1) \\ \sum_{m=0}^{M-1} B_m F_2(q, m, k_c b, k_c d) = 0, & q = 0,1,\ldots,(M-P-1) \end{cases} \quad (37)$$

At fixed number of partial modes M, the number of equations of the first type is defined by the angular widths ratio of the regions I and II (Fig. 2) as $P = \text{int}[\pi - 2\gamma)/\pi M]$, where integer part is rounded up or down [27].

The SLAE (37) can be rewritten in the matrix form by the formula (19), but the matrix $[F]$ elements are different:

$$F(i,j) = \begin{bmatrix} F_1(i, j, k_c a, k_c b, k_c d), & i = 0,1,\ldots,(P-1) \\ F_2((i-P), j, k_c b, k_c d), & i = P,(P+1),\ldots,(M-1) \end{bmatrix}.$$

The condition of nontrivial solution of the homogeneous SLAE (37) is the equality to zero of the matrix $[F]$ determinant. This condition defines the cutoff mode numbers of TE modes. The cutoff wave numbers calculated are to be substituted in homogeneous SLAE (37). The further solution of the TE modes problem is the same as the one described hereinbefore for the TEM mode except for the calculation of electric and magnetic fields components distributions. The distributions of longitudinal component of magnetic field H_z in the regions I and II (Fig. 2) can be defined by the formulas (20), (21). The magnetic field solution for the entire cross section of CQRW is found by using the symmetry or the antisymmetry properties of the TE modes. Transversal components of magnetic and electric fields are defined by using the formulas (38)–(41) connecting longitudinal and transversal field components (in which for the TE modes $E_z \equiv 0$):

$$E_r(r,\phi) = -\frac{i\beta}{k_c^2}\frac{\partial E_z(r,\phi)}{\partial r} - \frac{Z(f,k_c)}{k_c r}\frac{\partial H_z(r,\phi)}{\partial \phi}; \quad (38)$$

$$E_\phi(r,\phi) = -\frac{i\beta}{k_c^2 r}\frac{\partial E_z(r,\phi)}{\partial \phi} + \frac{Z(f,k_c)}{k_c}\frac{\partial H_z(r,\phi)}{\partial r}; \quad (39)$$

$$H_r(r,\phi) = -\frac{i\beta}{k_c^2}\frac{\partial H_z(r,\phi)}{\partial r} + \frac{Y(f,k_c)}{k_c r}\frac{\partial E_z(r,\phi)}{\partial \phi}; \quad (40)$$

$$H_\phi(r,\phi) = -\frac{i\beta}{k_c^2 r}\frac{\partial H_z(r,\phi)}{\partial \phi} - \frac{Y(f,k_c)}{k_c}\frac{\partial E_z(r,\phi)}{\partial r}, \quad (41)$$

where $Y(f,k_c) = 2\pi i f \varepsilon_a / k_c$; β denotes the longitudinal mode number of CQRW; i designates an imaginary unit; f is the operating frequency; ε_a defines the absolute permittivity of CQRW inner medium.

TE modes symmetric relative to the plane $\varphi = \pi/2$

In the regions I and II (Fig. 3), we represent the fields H_z and E_ϕ in the form of infinite sums (20)—(23) of the partial modes with unknown amplitudes and cutoff wave numbers. Each partial mode satisfies the Maxwell equations in the cylindrical coordinate system and boundary conditions at the two magnetic walls or at the magnetic and electric walls as well as at the perfectly conducting surfaces of CQRW, where $l_1(n) = 4\pi n/(\pi - 2\gamma)$, $l_2(m) = 4m + 2$ for the TE modes with antisymmetric EFD relative to the plane $\phi = \pi/4$ (for which this plane is the electric wall) or $l_1(n) = 2\pi(2n+1)/(\pi - 2\gamma)$, $l_2(m) = 4m + 4$ for the TE modes with symmetric EFD relative to the plane $\phi = \pi/4$ (for which this plane is the magnetic wall).

The boundary conditions at the interface between the regions I and II (Fig. 3) are as follows:

$$E_\phi^{II}(r = d,\ \phi \in [\gamma/2; \pi/4]) = = E_\phi^{I}(r = d,\ \phi \in [\gamma/2; \pi/4]); \quad (42)$$

$$H_z^{II}(r = d,\ \phi \in [\gamma/2; \pi/4]) = = H_z^{I}(r = d,\ \phi \in [\gamma/2; \pi/4]). \quad (43)$$

Besides, there is the following relation at the perfectly conducting surface of the ridge at $r = d$ and $\varphi \in [0; \gamma/2]$:

$$E_\phi^{II}(r = d,\ \phi \in [0;\ \gamma/2]) = 0. \quad (44)$$

Substituting (20)—(23) in (42)—(44) yields the results:

$$\sum_{m=0}^{\infty} B_m \sin[l_2(m)\phi] J'Y'[l_2(m), k_c b, k_c d] = = \sum_{n=0}^{\infty} A_n \cos[l_1(n)(\phi - \gamma/2)] J'Y'[l_1(n), k_c a, k_c d],$$
$$\varphi \in [\gamma/2; \pi/4]; \quad (45)$$

$$\sum_{m=0}^{\infty} B_m \sin[l_2(m)\phi] J'Y[l_2(m), k_c b, k_c d] = = \sum_{n=0}^{\infty} A_n \cos[l_1(n)(\phi - \gamma/2)] J'Y[l_1(n), k_c a, k_c d],$$
$$\varphi \in [\gamma/2; \pi/4]; \quad (46)$$

$$\sum_{m=0}^{\infty} B_m \sin[l_2(m)\phi] J'Y'[l_2(m), k_c b, k_c d] = 0,$$
$$\varphi \in [0; \gamma/2]. \quad (47)$$

Multiplying left and right parts of the equation (45) by the system of functions $\cos[l_1(p)(\phi - \gamma/2)]$, $p = 0,1,2,...$ and integrating the resulting relation at the interval $[\gamma/2; \pi/4]$, at which the system of these functions is orthogonal, we obtain:

$$\sum_{m=0}^{\infty} B_m I_1(p,m) J'Y'[l_2(m), k_c b, k_c d] = = A_p \frac{\pi - 2\gamma}{8}(1 + \delta_{p0}) J'Y'[l_1(p), k_c a, k_c d] \quad (48)$$

for the TE modes with antisymmetric EFD relative to the plane $\phi = \pi/4$;

$$\sum_{m=0}^{\infty} B_m I_1(p,m) J'Y'[l_2(m), k_c b, k_c d] = = A_p \frac{\pi - 2\gamma}{8} J'Y'[l_1(p), k_c a, k_c d] \quad (49)$$

for the TE modes with symmetric EFD relative to the plane $\phi = \pi/4$.

By using relations (48), (49), the amplitude of the p-th partial mode in the region I (Fig. 3) can be obtained:

$$A_p = \frac{8\sum_{m=0}^{\infty} B_m I_1(p,m) J'Y'[l_2(m), k_c b, k_c d]}{(\pi - 2\gamma)(1 + \delta_{p0}) J'Y'[l_1(p), k_c a, k_c d]}, \quad (50)$$

for the TE modes with antisymmetric EFD relative to the plane $\phi = \pi/4$;